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NUMERICAL SOLUTION OF N^{TH}-ORDER FUZZY INITIAL VALUE PROBLEMS BY NON-LINEAR TRAPEZOIDAL METHOD BASED ON LOGARITHMIC MEAN WITH STEP SIZE CONTROL

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ABSTRACT

In this paper, a numerical method for N^{th}- order Fuzzy Initial Value Problems (FIVP) based on Seikkala derivative of fuzzy process is studied. The non-linear trapezoidal method based on logarithmic mean is used to find the numerical solution of the FIVP and the convergence and stability of the method is given. This method is illustrated by solving second and third order FIVPs.

KEYWORDS: Triangular Fuzzy Number, N^{th}- Order Fuzzy Initial Value Problem, Non-Linear Trapezoidal Method, Logarithmic Mean, Step Size Control

1. INTRODUCTION

The research work on Fuzzy Differential Equations (FDEs) has been rapidly developing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh[8],it was followed up by Dubois and Prade [10] by using the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [27] and Goetschel and Voxman [17]. Kandel and Byatt [24] applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [22, 23], Seikkala [29], He and Yi [18]. The numerical methods for solving fuzzy differential equations are introduced by Abbasbandy et.al. and Allahviranloo et.al. in [2, 3]. Buckley and Feuring [7] introduced two analytical methods for solving N^{th} - order linear differential equations with fuzzy initial value conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function. Allahviranloo et.al [4, 5] proposed the methods for solving N^{th} – order fuzzy differential equations. Jayakumar et.al [21] used the Runge - Kutta Nystrom method for solving N^{th} – order fuzzy differential equations. Ivaz et.al [19] developed a numerical method namely trapezoidal rule for fuzzy differential equations and hybrid fuzzy differential equations. Gethsi Sharmila and Henry Amirtharaj [14,15]introduced the numerical solution of first-order fuzzy initial value problem by non-linear trapezoidal formulae based on variety of meansand developed a numerical algorithm for finding the solution of FIVPs by fourth order Runge-Kutta method based on Contra-harmonic mean. Gethsi Sharmila and Henry Amirtharaj [16] developed a numerical algorithm for finding the solution of N^{th}-order fuzzy initial value problems by fourth order Runge-Kutta method based on Centroidal Mean.

Non-linear trapezoidal formulae based on variety of means have found wide applications in the studies like mechanical vibrations, electrical circuits, planetary motions, etc. Several Researchers have applied non-linear trapezoidal formulae based on variety of means and analyze their error control and stability analysis. Necdet Bildik and Mustafa Inc[26] studied on the numerical solution of Initial value problems for non-linear trapezoidal formulae with different types.

D. J. Evans and B. B. Sanugi [11] compared the non-linear trapezoidal formulae using Euler, Harmonic mean, and logarithmic mean for solving IVPs Abdul-Majid Was was [30] made a comparison of modified Runge - Kutta formulae based on variety of means. K. Murugesan, et.al.[25] have done a comparison of extended Runge-Kutta formulae based on variety of means to solve systems of IVPs.

In engineering and physical problems, Trapezoidal rule is a simple and powerful method to solve numerically related ODEs. Trapezoidal rule has a higher convergence order in comparison to other one step methods, for instance, Euler method.

In this paper, the numerical methods to solve N^{th} - order linear fuzzy initial value problem is presented using the non-linear trapezoidal method based on logarithmic mean. The structure of the paper is organized as follows: In Section 2, we briefly present the fuzzy preliminaries are given. A fuzzy initial value problem is defined in Section 3. Trapezoidal rule and the non-linear trapezoidal formulae based on logarithmic mean for solving FIVPs are given in section 4. Convergence and stability of the mentioned method is proved in section 5. The step size control with the numerical method is discussed in section 6. The proposed algorithm is illustrated by solving the second and third order FIVPs in Section 7 and the conclusion is drawn in Section 8.

2. PRELIMINARIES

An arbitrary fuzzy number is represented by an ordered pair of functions

$\left(\underline{u}(r),\overline{u}(r)\right)$ for all $r\in[0,1]$, which satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1],
- $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1],
- $\underline{u}(r)\le\overline{u}(r),\ 0\le r\le 1.$

Let *E* be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $\alpha-$ level intervals.

Lemma 2.1

Let $\left[\underline{v}(\alpha),\overline{v}(\alpha)\right],\ \alpha\in(0,1]$ be a given family of non-empty intervals. If

$$(i)\ [\underline{v}(\alpha),\overline{v}(\alpha)]\supset[\underline{v}(\beta),\overline{v}(\beta)] \text{ for } 0<\alpha\le\beta, \text{ and } (ii)\ [\lim_{k\to\infty}\underline{v}(\alpha_k),\lim_{k\to\infty}\overline{v}(\alpha_k)]=[\underline{v}(\alpha),\overline{v}(\alpha)],$$

whenever (α_k) is a non-decreasing sequence converging to $\alpha\in(0,1]$, then the family $[\underline{v}(\alpha),\overline{v}(\alpha)],\ \alpha\in(0,1]$ represent the $\alpha-$ level set of fuzzy number v in E. Conversely if $[\underline{v}(\alpha),\overline{v}(\alpha)],\ \alpha\in(0,1]$, are $\alpha-$ level set of fuzzy number $v\in E$ then the conditions (i) and (ii) hold true.

Definition 2.1

Let I be a real interval. A mapping $v: I \to E$ is called a fuzzy process and denote the $\alpha-$ level set by $[v(t)]_\alpha = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$. The Seikkala derivative $v'(t)$ of v is defined by $[v'(t)]_\alpha = [\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)]$, provided that is a equation defines a fuzzy number $v'(t) \in E$

Definition 2.2

Suppose u and v are fuzzy sets in *E*. Then their Hausdorff

$$D: E \times E \to R_+ \cup \{0\},\ D(u,v) = \sup_{\alpha \in [0,\, 1]} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|,\ \left|\overline{u}(\alpha) - \overline{v}(\alpha)\right|\right\},$$

i.e $D(u,v)$ is maximal distance between α level sets of u and v .

3. FUZZY INITIAL VALUE PROBLEM

Now we consider the initial value problem

$$\left\{x^{(n)}(t) = \psi(t, x, x', ..., x^{(n-1)}),\ x(0) = a_1, ...,\ x^{(n-1)}(0) = a_n, \right. \tag{3.1}$$

where ψ is a continuous mapping from $R_+ \times R^n$ into R and $a_i (0 \le i \le n)$ are fuzzy numbers in E. The mentioned N^{th} - order fuzzy differential equation by changing variables

$$y_1(t) = x(t),\ y_2(t) = x'(t), ...,\ y_n(t) = x^{(n-1)}(t),$$

converts to the following fuzzy system

$$\begin{cases} y_1'(t) = f_1(t, y_1, ...,\ y_n), \\ \qquad \vdots \\ y_n'(t) = f_n(t, y_1, ...,\ y_n), \\ y_1(0) = y_1^{[0]} = a_1, ...,\ y_n(0) = y_n^{[0]} = a_n, \end{cases} \tag{3.2}$$

where $f_i\ (1 \le \mathrm{i} \le \mathrm{n})$ are continuous mapping from $R_+ \times R^n$ into R and $y_i^{[0]}$ are fuzzy numbers in E with α - level intervals.

$$[y_i^{[0]}]_\alpha = [\underline{y}_i^{[0]}(\alpha), \overline{y}_i^{[0]}(\alpha)] \text{ for i = 1,..., n and } 0 \le \alpha \le 1$$

We call $y = (y_1, ...,\ y_n)^T$ is a fuzzy solution of (3.2) on an interval I, if

$$\underline{y}'_i(t,\alpha) = \min\left\{ f_i(t,u_1,..., \mathrm{u}_n);\ \mathrm{u}_j \in [\underline{y}_j(t,\alpha),\ \overline{y}_j(t,\alpha)] \right\} = \underline{f}_i(t, y(t,\alpha)),$$

$$\overline{y}'_i(t,\alpha) = \max\left\{ f_i(t,u_1,..., \mathrm{u}_n);\ \mathrm{u}_j \in [\underline{y}_j(t,\alpha),\ \overline{y}_j(t,\alpha)] \right\} = \overline{f}_i(t, y(t,\alpha)), \tag{3.3}$$

and

$$\underline{y}_i(0,\alpha) = \underline{y}_i^{[0]}(\alpha),\ \overline{y}_i(0,\alpha) = \overline{y}_i^{[0]}(\alpha) \tag{3.4}$$

Thus for fixed α we have a system of initial value problem in R^{2n} . If we can solve it (uniquely), we have only to verify that the intervals, $[\underline{y}_j(t,\alpha), \overline{y}_j(t,\alpha)]$ define a fuzzy number $y_i(t) \in E.$ Now let $\underline{y}^{[0]}(\alpha) = (\underline{y}_1^{[0]}(\alpha),..., \underline{y}_n^{[0]}(\alpha))^T$ and $\overline{y}^{[0]}(\alpha) = (\overline{y}_1^{[0]}(\alpha),..., \overline{y}_n^{[0]}(\alpha))^T$ with respect to the above mentioned indicators, system (3.2) can be written as with assumption

$$\begin{cases} y'(t) = F(t, y(t)), \\ y(0) = y^{[0]} \in E^n. \end{cases} \tag{3.5}$$

With assumption

$$y(t,\alpha) = [\underline{y}(t,\alpha), \overline{y}(t,\alpha)] \ and \ \mathrm{y}'(t,\alpha) = [\underline{y}'(t,\alpha), \overline{y}'(t,\alpha)] \tag{3.6}$$

where

$$\overline{y}(t,\alpha) = [\overline{y}(t,\alpha),...,\ \overline{y}(t,\alpha)]^T, \tag{3.7}$$

$$\underline{y}'(t,\alpha) = [\underline{y}'(t,\alpha),...,\ \underline{y}'(t,\alpha)]^T, \tag{3.8}$$

$$\overline{y'}(t,\alpha) = [\overline{y'}(t,\alpha),...,\ \overline{y'}(t,\alpha)]^T, \tag{3.9}$$

and with assumption $F(t, y(t,\alpha)) = [\underline{F}(t, y(t,\alpha))), \overline{F}(t, y(t,\alpha))]$, where

$$\underline{F}(t, y(t,\alpha)) = [\underline{f}_1(t, y(t,\alpha))),...,\ \underline{f}_n(t, y(t,\alpha))]^T, \tag{3.10}$$

$$\overline{F}(t, y(t,\alpha)) = [\overline{f}_1(t, y(t,\alpha))),...,\ \overline{f}_n(t, y(t,\alpha))]^T, \tag{3.11}$$

y (t) is a fuzzy solution of (3.5) on an interval *I* for all $\alpha \in (0,1]$, if

$$\begin{cases} \underline{y}'(t,\alpha)) = \underline{F}(t, y(t,\alpha)); \\ \overline{y}'(t,\alpha)) = \overline{F}(t, y(t,\alpha)) \\ \underline{y}(0,\alpha) = \underline{y}^{[0]}(\alpha),\ \overline{y}(0,\alpha) = \overline{y}^{[0]}(\alpha) \end{cases} \tag{3.12}$$

or

$$\begin{cases} y'(t,\alpha)) = F(t, y(t,\alpha)), \\ y(0,\alpha) = y^{[0]}(\alpha). \end{cases} \tag{3.13}$$

Now we show that under the assumptions for functions $f_{i,}$ for i=1,..., n how we can obtain a unique fuzzy solution for system (3.2).

Theorem 3.1

If $f_i(t,u_1,\ldots,\ \mathrm{u}_n)$ for $i=1,\ldots,\ \mathrm{n}$ are continuous function of *t* and satisfies the Lipschitz condition in $u = \ (u_1,\ldots,\ \mathrm{u}_n)^T$ in the region $D = \{t, u | t \in I = [0,1], -\infty < u_i < \infty \ for\ \mathrm{i}=1,\ldots,\ \mathrm{n}\}$ with constant L_i then the initial value problem (3.2) has a unique fuzzy solution in each case.

Proof. Refer [1]

4. THE NON-LINEAR TRAPEZOIDAL METHOD BASED ON LOGARITHMIC MEAN TO SOLVE FUZZY INITIAL VALUE PROBLEM

4.1 Preliminary Notes

4.1.1 Non-Linear Trapezoidal Formulae

Increment Function of Logarithmic Mean

We consider the initial value problem for a first order ordinary differential equation

$$y' = f(t,y), y(t_0) = y_0 \tag{4.1}$$

The function $f(t,y)$ is to be continuous for all (t,y)in some domain D of the xy-plane, and (x_0,y_0) is a point in D where D={(t, y)/a ≤t ≤ b, -∞<y<∞}

Suppose we want to compute an approximate value $y(t)$ (t being some real number) which will be existent and unique. To compute this numerical approximation, we use one-step method of the form

$$y_{n+1} = y_n + h\Phi(t_n, t_{n+1}, y_n, y_{n+1}, h), n = 0,1, \ldots k-1. \tag{4.2}$$

with an increment function Φ of certain special type. Here, n is a (user-chosen) arbitrary step size parameter, and put

$$h = \frac{t_n - t_0}{n} \text{and} t_n = t_0 + nh, forn = 1, \ldots, k. \tag{4.3}$$

Finally, we take the value $y(t:h) = y_n$as an approximation of the desired result$y(t)$.

One of the most popular choices for the increment function Φ is the arithmetic mean of the two values $f_n = f(t_n, y_n)$and $f_{n+1} = f(t_{n+1}, y_{n+1})$

i.e. $\Phi^A(t_n, t_{n+1}, y_n, y_{n+1}, h) = \frac{f_n+f_{n+1}}{2}$which yields the **trapezoidal rule**. This is a one-step implicit finite-difference method which is frequently employed for the numerical solution of (4.1). In fact it is well known that it is the most accurate A- stable multi-step method [9].

However, it has been pointed out by D. J. Evans and B.B. Sanugi that taking the arithmetic mean is not in all cases the best choice, and consequently the use of other means, e.g. Geometric, Harmonic or logarithmic ones has been studied. The resulting methods were denoted as **non-linear trapezoidal formulae.**

Now, let Φ denote one of the increment function, defined through non- linear logarithmic mean:

$\Phi^L(t_n, t_{n+1}, y_n, y_{n+1}, h)$= $\left(\frac{f_{n+1}-f_n}{In\left(\frac{f_{n+1}}{f_n}\right)}\right)$ (Logarithmic)

The standard Trapezoidal formula based on arithmetic mean (AM) formula for solving initial value problems of the form $y' = f(t, y), y(t_0) = y_0$ is written in the form

$y_{n+1=}y_n + h\left(\frac{f_n+f_{n+1}}{2}\right)$, where h is the mean length in the local t-direction.

The Non-linear Trapezoidal formula based on logarithmic mean is : $y_{n+1} = y_n + h\left(\frac{f_{n+1} - f_n}{\ln\left(\frac{f_{n+1}}{f}\right)}\right)$,

its Local Truncation Error is $y(t_{n+1}) - y_{n+1} = h^3\left(\frac{-y_n'''}{12} - \frac{y_n''^2}{12y_n'}\right) + o(h^4)$

and Stability of this method is: L- stable.

4.2 Non-Linear Trapezoidal Method Based on Logarithmic Mean Formula for Solving System of IVPs

Evans and Sanugi[11] have developed a new Non-linear trapezoidal method based on Logarithmic Mean to solve first order equation and it is to be noted that the Logarithmic mean for the two positive quantities f_i and f_{i+1} is defined as

$$\frac{f_{i+1} - f_i}{\ln\left(\frac{f_{i+1}}{f_i}\right)}. \tag{4.4}$$

In conjunction with the logarithmic mean given by (4.1) we may now form the corresponding integration formula to solve N^{th} - order IVPs given by

$$y_{n+1j} = y_{nj} + h\left[\frac{y_{n+1j}{}' - y_{nj}{}'}{\ln\left(\dfrac{y_{n+1j}{}'}{y_{nj}{}'}\right)}\right], \quad 1 \le j \le m$$

Since it is of implicit nature,to find the value of $y_{n+1j}{}'$, one can use the classical fourth order Runge- Kutta formula.

4.2Non-Linear Trapezoidal Method Based on Logrithmic Mean for Solving N^th^-Order FIVPs

We consider the N^{th}-order fuzzy initial value problem (3.2) with the unique solution $y = (y_{1,},\ldots,\ y_n)^T \in E^n$. For finding an approximate solution of (3.2) with the trapezoidal rule based on Arithmetic mean, we first define the trapezoidal rule as follows:

$$\begin{aligned}
&\underline{y}^{\alpha}_{p+1j} = \underline{y}^{\alpha}_{pj} + \tfrac{h}{2}\left[\underline{f_j}^{\alpha}(t_p, y_p) + \underline{f_j}^{\alpha}(t_{p+1}, y_{p+1})\right] \\
&\overline{y}^{\alpha}_{p+1j} = \overline{y}^{\alpha}_{pj} + \tfrac{h}{2}\left[\overline{f}_j^{\alpha}(t_p, y_p) + \overline{f}_j(t_{p+1}, y_{p+1})\right] \qquad (4.5)\\
&\underline{y}_p{}^{\alpha} = \underline{\gamma}, \overline{y}_p{}^{\alpha} = \overline{\gamma}, \text{for } 0 \le \text{p} \le \text{N}.
\end{aligned}$$

Formulae for solving N^{th}- order fuzzy initial value problem by using non-linear Trapezoidal formulae based on logarithmic mean is given as follows:

$$\text{Logarithmic Mean: } \underline{y}^{\alpha}_{p+1j} = \underline{y}^{\alpha}_{pj} + h\left[\frac{\underline{f_j}^{\alpha}(t_{p+1},y_{p+1}) - \underline{f_j}^{\alpha}(t_p,y_p)}{ln\left(\dfrac{\underline{f_j}^{\alpha}(t_{p+1},y_{p+1})}{\underline{f_j}^{\alpha}(t_p,y_p)}\right)}\right]$$

$$\overline{y}^{\alpha}_{p+1j} = \overline{y}^{\alpha}_{pj} + h\left[\frac{\overline{f}_j^{\alpha}(t_{p+1},y_{p+1}) - \overline{f}_j^{\alpha}(t_p,y_p)}{ln\left(\dfrac{\overline{f}_j^{\alpha}(t_{p+1},y_{p+1})}{\overline{f}_j^{\alpha}(t_p,y_p)}\right)}\right] \qquad (4.6)$$

$$\underline{y}_p{}^{\alpha} = \underline{\gamma}, \overline{y}_p{}^{\alpha} = \overline{\gamma}, \text{for } 0 \le \text{p} \le \text{N}.$$

The classical fourth order Runge-Kutta method [1] for solving N^{th} – order fuzzy initial value problem can be used to find the value of $\underline{y}_{p+1j}, \overline{y}_{p+1j}$, since this formula is of implicit nature.

5. CONVERGENCE AND STABILITY [1]

Definition 5.1

A one-step method for approximating the solution of a differential equation (5.1)

Which F is a N[th]- ordered as follow $f = (f_1, ..., f_n)^T$ and $f_i : \mathrm{R}_+ \times R^n \to R (1 \le i \le n)$, is a method which can be written in the form

$$y^{[n+1]} = y^{[n]} + h \ \psi \ (t_n, y^n, h), \tag{5.2}$$

where the increment function ψ is determined by $\boldsymbol{F}$ and is a function of $t_n, y^{[n]}$ and h only.

Theorem 5.1

If $\psi \ (t, y, h)$ satisfies a Lipschitz condition in $\boldsymbol{y}$, then the method given by (5.2) is stable.

Theorem 5.2

In relation (3.5), if $\boldsymbol{F}(t, \boldsymbol{y})$ satisfies a Lipschitz condition in $\boldsymbol{y}$, then the method given by (4.6) is stable.

Theorem 5.3

If $y^{[m+1]}(\alpha) = y^{[m]}(\alpha) + h \ \psi \ (t_m, y^m(\alpha), h), \ \ \mathrm{y}^{[0]}(\alpha) = Y^{[m]}(\alpha)$ (5.3)

where $\psi \ (t_m, y^m(\alpha), h) = [\psi_1 \ (t_m, y^m(\alpha), h), \ \psi_2 \ (t_m, y^m(\alpha), h)]$ is a numerical method for approximation of differential equation (3.13), and ψ_1 and ψ_2 are continuous in t, y, h for $0 \le t \le T, \ 0 \le h \le h_0$ and all $\boldsymbol{y}$, and if they satisfy a Lipschitz condition in the region $D = \{t, u, v, h) | 0 \le t \le T, \ -\infty < u_i \le v_i, \ -\infty < v_i \le +\infty, \ 0 \le h \le h_0 \ \text{i=1, ..., n}\}$, necessary and sufficient conditions for convergence above mentioned method is

$$\psi(t, y(t, \alpha), h) = F(t, y(t, \alpha)). \tag{5.4}$$

6. STEP SIZE CONTROL

An important consideration in using the discrete method is that of estimating the local error and of selecting the proper step size to achieve a required accuracy. There is no reason why the step size h needs to be kept fixed over the entire interval. Estimating the accuracy using different fixed step sizes may be very inefficient. In this section we will examine methods for estimating the local error and for varying the step size according to some error criterion.

The method is based on interval halving. Let us assume that we are using the method of order p and that we have arrived at a point t_n with $h = \mathrm{t}_\mathrm{n}$ - $\mathrm{t}_\mathrm{n-1}$. We now integrate from t_n to $\mathrm{t}_\mathrm{n-1} = \mathrm{t}_\mathrm{n} \ + \mathrm{h}$ twice, once using the current step h and again two steps of length $h/2$. We will thus obtain two estimates $y_h(t_{n+1})$ and $y_{h/2}(t_{n+1})$ of the value of $y(t)$ at $\mathrm{t} = \mathrm{t}_\mathrm{n-1}$ and a comparison of these two estimates will yield an estimate of the error. To derive the estimate of order p has a local asymptotic error expansion of the form.

$$y_h(t_n + mh) = y(t_n + mh) + C(t_n + mh)h^p + 0(h^{p+1}) \tag{6.1}$$

Here $y_h(t_n+mh)$ denotes the approximation to the solution y(x) at the point $x = \mathrm{t_n} + \mathrm{mh}$. Further the constant $C(t_n+mh)$ does not depend on h, though it does depend on f(t,y) and on the point $t = \mathrm{t_n} + \mathrm{mh}$. Therefore

$$y_h(t_{n+1}) = y(t_{n+1}) + C(t_{n+1})h^p + 0(h^{p+1}) \tag{6.2}$$

$$y_{h/2}(t_{n+1}) = y(t_{n+1}) + C(t_{n+1})(h/2)^p + 0(h^{p+1}) \tag{6.3}$$

on subtracting (6.2) from (6.3) we find that the principal part of the error in (6.3) can be estimated as

$$C_n\left(\frac{h}{2}\right)^p \approx \frac{y_{h/2}(t_{n+1}) - y_h(t_{n+1})}{1-2^p}$$

The quantity

$$D_n \approx \frac{\left|y_{h/2}(t_{n+1}) - y_h(t_{n+1})\right|}{2^p \text{ -1}} \tag{6.4}$$

Thus provides us with a computable estimate of the error D_n in the approximation $y_{h/2}(t_{n+1})$ and it can be used to help us decided whether the step h being used is just right, too big, or too small.

For the Fuzzy Initial Value Problems, the computable estimate of the error $\left(\underline{D}_n \approx \frac{\left|\underline{y}_{h/2}(t_{n+1}) - \underline{y}_h(t_{n+1})\right|}{2^p \text{ -1}}, \overline{D}_n \approx \frac{\left|\overline{y}_{h/2}(t_{n+1}) - \overline{y}_h(t_{n+1})\right|}{2^p \text{ -1}}\right)$ in the approximation $\left(\underline{y}_{h/2}(t_{n+1}), \overline{y}_{h/2}(t_{n+1})\right)$.

Suppose now that we are given some local error tolerance ε and that we wish to keep the estimated error $\left(\underline{D}_n, \overline{D}_n\right)$ below the local error tolerance per unit step, i.e., we want

$$\left(\underline{D}_n, \overline{D}_n\right) \leq \varepsilon \text{ h} \tag{6.5}$$

Assume that we have computed $\left(\underline{y}_h(t_{n+1}), \overline{y}_h(t_{n+1})\right)$, $\left(\underline{y}_{h/2}(t_{n+1}), \overline{y}_{h/2}(t_{n+1})\right)$ and $\left(\underline{D}_n, \overline{D}_n\right)$. We must now decide on whether to accept the value $\left(\underline{y}_{h/2}(t_{n+1}), \overline{y}_{h/2}(t_{n+1})\right)$ and on what step h to use for the next integration. From the given error tolerance ε, we compute a lower error bound $\varepsilon' < \varepsilon$. we have the following possibilities:

- $$\varepsilon' < \max\left(\frac{\underline{\mathrm{D}}_\mathrm{n}}{\mathrm{h}}, \frac{\overline{\mathrm{D}}_\mathrm{n}}{\mathrm{h}}\right) < \varepsilon$$

In this case we accept the value $y_{h/2}(t_{n+1})$ and continue the integration from t_{n+1} using the same step size h

- $$\max\left(\frac{\underline{\mathrm{D}}_\mathrm{n}}{\mathrm{h}}, \frac{\overline{\mathrm{D}}_\mathrm{n}}{\mathrm{h}}\right) > \varepsilon$$

In this case the error is too large, hence we must reduce h - say to $h/2$ - and integrate again from the point $t = t_n$

- $\max\left(\frac{\underline{D}_n}{h}, \frac{\overline{D}_n}{h}\right) < \varepsilon'$

In this case we are getting more accuracy than required. We accept the value $\left(\underline{y}_{h/2}(t_{n+1}), \overline{y}_{h/2}(t_{n+1})\right)$ replace h - by say to $2h$ - and integrate from t_{n+1}.

If we restrict the interval step size to halving or doubling, then the lower bound ε' can be set to $\varepsilon' = \varepsilon^{p+1}$.

For a p^{th} order method since halving the step size reduces the error by approximately

$1/2^{p+1}$. Hence $\varepsilon' = \varepsilon / 32$ for p = 4. Actually it is not advisable to change the step size too often, and to be safe one might use $\varepsilon' = \varepsilon / 50$.

7. NUMERICAL RESULTS

Example 7.1

Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{cases} y''(t) - 4y'(t) + 4y(t) = 0 & (t \geq 0) \\ y(0) = (2+\alpha, 4-\alpha) \\ y'(0) = (5+\alpha, 7-\alpha) \end{cases}$$

The exact solution is as follows:

$$\underline{y}(t,r) = (2+r)e^{2t} + (1-r)te^{2t})$$
$$\overline{y}(t,r) = (4-r)e^{2t} + (r-1)te^{2t})$$

Here our proposed method i.e. nonlinear trapezoidal method based on logarithmic mean is used to solve the above problem. Interval halving method is used to control the step by setting some error tolerance $\varepsilon = 0.0001$, and a lower error bound $\varepsilon' = 0.000002$.

The following table 1 provides the computed ratio between the estimate of error and the value of h when the values of r = 0, 0.2, 0.4, 0.6, 0.8 and 1 with t = 1. To keep the estimate error $\left(\underline{D}_n, \overline{D}_n\right)$ below the local error tolerance per unit step, we have to check $\frac{\underline{D}_n}{h} \leq \varepsilon$ and $\frac{\overline{D}_n}{h} \leq \varepsilon$ for the lower and upper cut values

And the following figure is the graphical representation of the approximate value with the r – level cut when t = 1.

Table 1: The Ratio between the Estimate of Error and H by Using the Interval Halving Method for T = 1

H	**r =0**		**r =0.2**		**r =0.4**	
	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$
0.5	0.068821333	0.050567867	0.0570944	0.038378667	0.0453284	0.0262544
0.25	0.1108176	0.109466133	0.089005867	0.0871872	0.067150667	0.06497413
0.125	0.2399344	0.2379376	0.1918256	0.1904544	0.1437904	0.14290987
0.0625	0.410944	0.4065312	0.328440533	0.325618133	0.246090667	0.24449813
0.03125	0.567025067	0.562382933	0.453314133	0.450365867	0.3397504	0.33810347
H	r = 0.6		r = 0.8		r = 1.0	
	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$
0.5	0.03352027	0.014191067	0.021667333	0.002185333	0.009766533	0.009766533
0.25	0.04525147	0.0428224	0.023306933	0.020727467	0.001314667	0.001314667
0.125	0.0958288	0.095300267	0.047939733	0.047622933	1.23E-04	1.23E-04
0.0625	0.16389867	0.163181867	0.081869867	0.081677867	0.0000096	0.0000096
0.03125	0.22633813	0.225608533	0.113083733	0.112900267	0	0

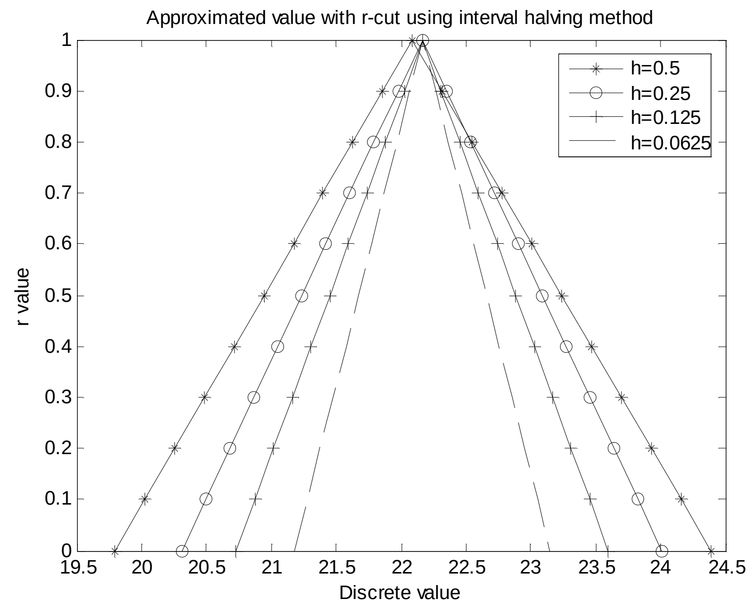

Figure 1

Example 7.2

Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{cases} y'''(t) = 2y''(t) + 3y'(t) \qquad (0 \le \mathrm{t} \le 1) \\ y(0) = (3+\alpha, 5-\alpha) \\ y'(0) = (-3+\alpha, -1-\alpha) \\ y''(0) = (8+\alpha, 10-\alpha) \end{cases}$$

The eigen value-eigenvector solution is as follows:

$$y(t,r)=(-\frac{1}{3}+\frac{7}{12}e^{3t}+(\frac{11}{4}+r)e^{-t},-\frac{1}{3}+\frac{7}{12}e^{3t}+(\frac{19}{4}-r)e^{-t}).$$

The following table 2 gives the results using the proposed method.

Table 2: Approximate, Exact and Absolute Error Values When r=1 and t=1 Using Step Size Control

h	$\underline{y}$	$\underline{Y}$	Abs.Error of $\underline{y}$	$\overline{y}$	$\overline{Y}$	Abs.Error of $\overline{y}$
0.5	12.21969	12.76278	1.20E+00	12.21969	12.76278	1.20E+00
0.25	12.70519	12.76278	2.11E-01	12.70519	12.76278	2.11E-01
0.125	12.73719	12.76278	6.66E-02	12.73719	12.76278	6.66E-02
0.0625	12.75128	12.76278	1.42E-02	12.75128	12.76278	1.42E-02
0.03125	12.75985	12.76278	3.91E-03	12.75985	12.76278	3.91E-03
0.015625	12.76207	12.76278	1.07E-03	12.76207	12.76278	1.07E-03
0.007813	12.76264	12.76278	2.76E-04	12.76264	12.76278	2.76E-04
0.003906	12.76276	12.76278	5.47E-05	12.76276	12.76278	5.47E-05
0.001953	12.76277	12.76278	1.65E-05	12.76277	12.76278	1.65E-05
0.000977	12.76278	12.76278	3.88E-06	12.76278	12.76278	3.88E-06
0.000488	12.76278	12.76278	1.06E-06	12.76278	12.76278	1.06E-06
0.000244	12.76278	12.76278	2.68E-07	12.76278	12.76278	2.68E-07
0.000122	12.76278	12.76278	4.61E-08	12.76278	12.76278	4.61E-08

By Setting some local error tolerance $\varepsilon = 0.0001$, and a lower error bound $\varepsilon' = 0.000002$,the ratio of estimate of error $\left(\underline{D_n}, \overline{D_n}\right)$ and the value of h are computed and listed in table 3 when the values of r = 1 and t = 1.

Table 3: The Ratio between the Estimate of Error D_n and h by Using the Interval Halving Method for r = 1 and t = 1 for Example 7.2

h	$\frac{\underline{D_n}}{h}$	$\frac{\overline{D_n}}{h}$
0.5	0.064732667	0.064732667
0.25	0.008534667	0.008534667
0.125	0.007513067	0.007513067
0.0625	0.0091424	0.0091424
0.03125	0.004746667	0.004746667
0.015625	0.002397867	0.002397867
0.007813	0.001075131	0.001075131
0.003906	1.71E-04	1.71E-04
0.001953	1.37E-04	1.37E-04
0.000977	1.36E-04	1.36E-04
0.000488	1.37E-04	1.37E-04
0.000244	0.00E+00	0.00E+00

8. CONCLUSIONS

In this paper, a numerical method for solving N^{th}- order fuzzy initial value problem is presented. In this method N^{th}- order fuzzy linear differential equation is converted to a fuzzy system which will be solved with the non-linear trapezoidal method based on logarithmic mean. For the numerical example 7.1, using the method of interval halving and by comparing the ratio between the estimate of error and the value of h at r=1 and t=1from the table 1 and the approximate value from the figure 1, it is concluded that the proposed method gives the required accuracy when the value of h = 0.0625for solving the second order FIVP. From the tables 2 and 3 of example 2, it is concluded that the proposed method with step size control works well to get the desired accuracy in solving the N^{th}- order FIVPs. By using interval halving in step size control the accuracy for the example 7.2 is obtained at h=0.000244which is achieved using the software MATLAB.

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